Integral Calculus

Integral calculus is the process of finding the function itself when its derivative is known. This process is called *integration*.

Suppose we are given the relation $\frac{dy}{dx} = 3x^2$ and we are asked to find y.

Using our knowledge with derivatives we can say that $y = x^3$ because we know that

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

However, the derivatives of other functions such as $y = x^3 + 2,$ $y = x^3 - 7,$ $y = x^3 + \sqrt{5}$ also equal $3x^2$. So we can say that the solution of this differential equation is

$y = x^3 + C$ C is a constant (constant of integration)

This differential equation can also be written as $dy = 3x^2 dx$

We say that dy is the differential of y in terms of x and dx, and dx is the differential of x.

The symbol $\int$ denotes integration. The integral $\int f(x)dx$ is an indefinite integral and $\int_a^b f(x)dx$ is a definite integral.

When we evaluate indefinite integrals, we must always add the constant of integration C.

Indefinite Integrals

We have some rules that make solving integrals easier.

1. The integral of the differential of a function x is x plus an arbitrary constant C.

$$\int dx = x + C$$

2. A constant, say a, may be written in front of the integral sign.

$$\int a\,dx = a\int dx$$

3. The integral of the sum of two or more differentials is the sum of their integrals.

$$\int (dx_1 + dx_2 + \ldots + dx_n) = \int dx_1 + \int dx_2 + \ldots + \int dx_n$$

4. If $n \neq -1$, the integral of $x^n dx$ is obtained by adding one to the exponent and dividing by the new exponent.

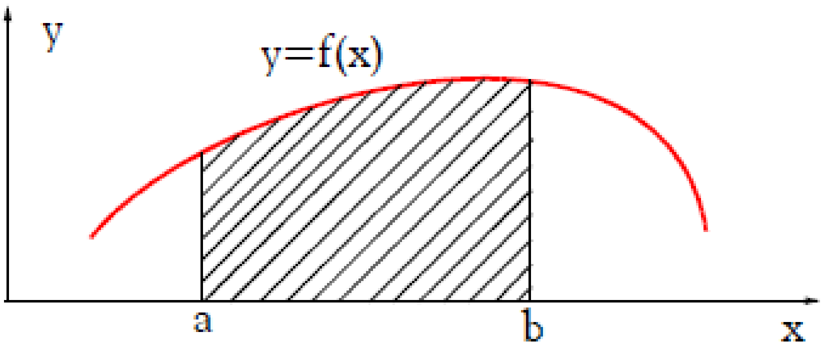
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

TABLE 5.1 Differentials and their integral counterparts

	Differentials	***Integrals***
1	$du = \frac{du}{dx}dx$	$\int du = u + C$
2	$d(au) = adu$	$\int adx = a\int dx$
3	$d(x_1 + x_2 + \ldots + x_n) = dx_1 + dx_2 + \ldots + x_n$	$\int(dx_1 + dx_2 + \ldots + dx_n) = \int dx_1 + \int dx_2 + \ldots + \int dx_n$
4	$d(x^n) = nx^{n-1}dx$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$
5	$d(\ln x) = \frac{dx}{x}$	$\int \frac{dx}{x} = \|\ln x\| + C$
6	$d(e^x) = e^x dx$	$\int e^x dx = e^x + C$
7	$d(a^x) = a^x \ln a dx$	$\int a^x dx = \frac{a^x}{\ln a} + C$
8	$d(\sin x) = \cos x dx$	$\int \cos x dx = \sin x + C$
9	$d(\cos x) = -\sin x dx$	$\int \sin x dx = -\cos x + C$
10	$d(\tan x) = \sec^2 x dx$	$\int \sec^2 x dx = \tan x + C$

Definite Integrals

The area under the curve y = f(x) between points a and b can be found from the integral



$$\boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)}$$

F(x) is the integral of f(x)dx

$F(x)\Big|_a^b$ means that we must first replace x with the upper value b, that is, we set x = b to obtain F(b), and from it we subtract the value F(a), which is obtained by setting x = a.

b = upper limit of integration
a = lower limit of integration

Note that the constant C cancels out so we can ignore it.

48.10 Find the area bounded by the parabola $y = 8 + 2x - x^2$ and the x axis.

The x intercepts are $x = -2$ and $x = 4$; $y \geq 0$ on the interval $-2 \leq x \leq 4$. See Fig. 48-4. Hence

$$A = \int_{-2}^{4} (8 + 2x - x^2)\,dx = \left(8x + x^2 - \frac{x^3}{3}\right)\Bigg|_{-2}^{4}$$
$$= \left(8 \cdot 4 + 4^2 - \frac{4^3}{3}\right) - \left[8(-2) + (-2)^2 - \frac{(-2)^3}{3}\right] = 36 \text{ sq units}$$

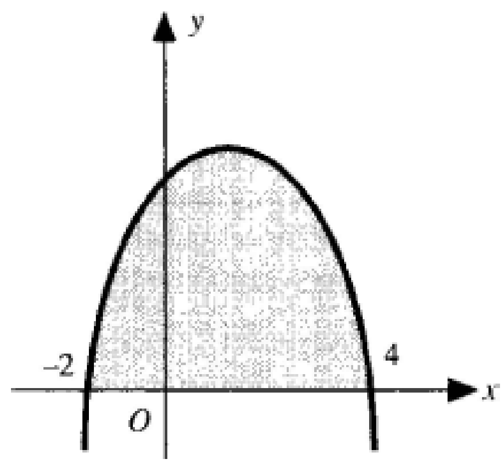


Fig. 48-4

48.12 Find the area bounded by the curve $y = x^3 - 9x$, the x axis, and the ordinates $x = -2$ and $x = 4$.

The purpose of this problem is to show that the required area is *not* given by $\int_{-2}^{4}(x^3 - 9x)\,dx$.

From Fig. 48-6, we note that y changes sign at $x = 0$ and at $x = 3$. The required area consists of three pieces, the individual areas being given, apart from sign, by

$$A_1 = \int_{-2}^{0}(x^3 - 9x)\ dx = \left(\tfrac{1}{4}x^4 - \tfrac{9}{2}x^2\right)\Big|_{2}^{0} = 0 - (4 - 18) = 14$$

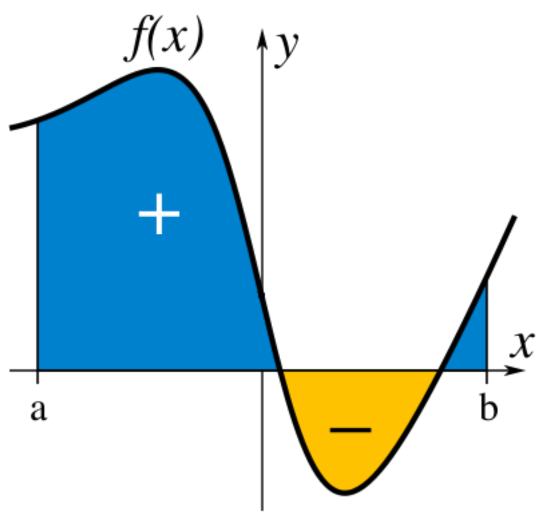
$$A_2 = \int_{0}^{3}(x^3 - 9x)\ dx = \left(\tfrac{1}{4}x^4 - \tfrac{9}{2}x^2\right)\Big|_{0}^{3} = \left(\tfrac{81}{4} - \tfrac{81}{2}\right) - 0 = -\tfrac{81}{4}$$

$$A_3 = \int_{3}^{4}(x^3 - 9x)\ dx = \left(\tfrac{1}{4}x^4 - \tfrac{9}{2}x^2\right)\Big|_{3}^{4} = (64 - 72) - \left(\tfrac{81}{4} - \tfrac{81}{2}\right) = \tfrac{49}{4}$$

Thus, $A = A_1 - A_2 + A_3 = 14 + \frac{81}{4} + \frac{49}{4} = \frac{93}{2}$ sq units.

Note that $\int_{-2}^{4}(x^3 - 9x)\ dx = 6 < A_1$, an absurd result.

Integration of the function between a and b gives the area above the curve minus the area below the curve i.e. the area in blue.



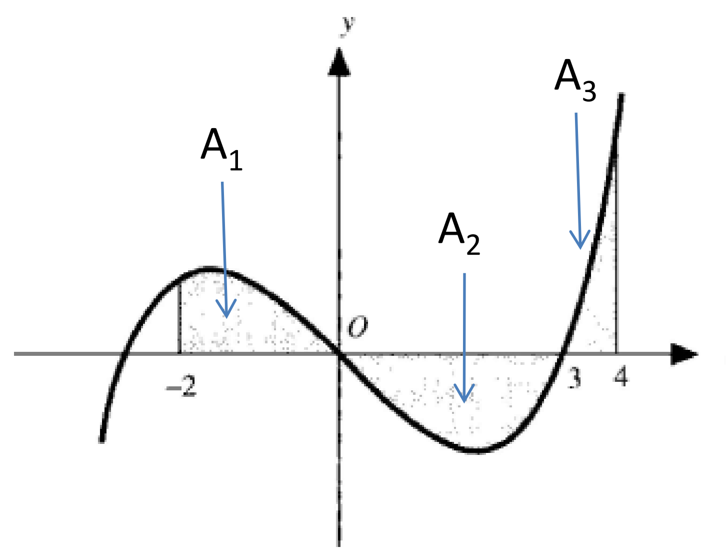


Fig. 48-6

Exercises

1. Evaluate the following indefinite integrals

a. $\int 3x^2 dx$ b. $\int (5x^3 + 6x^2 - 12)dx$ c. $\int 3e^{2x} dx$

2. Evaluate the following definite integrals, that is, find the area under the curve for the lower and upper limits of integration.

a. $\int_1^2 (2x^5 - 6x^3 + 3x)dx$ b. $\int_3^5 (3x^2 + 6x - 8)dx$

The Substitution Rule

Suppose that we want to find $\int (\cos(x^2) \cdot x) dx.$

Since sin(x) differentiates to cos(x), as a first guess we might try differentiating the function sin(x²).

By the Chain Rule

$$y = \sin\left(x^2\right) \qquad \text{Let } x^2 = u$$

$$y = \sin u \qquad \frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot 2x = 2x \cos\left(x^2\right)$$

We can remove the factor of 2 by dividing by a constant.

$$\frac{d}{dx}\frac{\sin(x^2)}{2} = \frac{1}{2} \cdot \frac{d}{dx}\sin(x^2) = \frac{1}{2} \cdot 2\cos(x^2)x = x\cos(x^2) = f(x).$$

So, $\int x\cos(x^2)dx = \frac{\sin(x^2)}{2} + C.$

This technique will work for more general integrands

Substitution rule for indefinite integrals : for a function f(u), where u is a function of x i.e. u(x) and u is differentiable:

$$\int f(u(x))\frac{du}{dx}dx = \int f(u)du.$$

Substitution rule for definite integrals: assume u is differentiable and suppose c = u(a) and d = u(b).

$$\int_a^b f(u(x))\frac{du}{dx}dx = \int_c^d f(u)du.$$

Notice that it looks like you can *cancel* in the expression $\frac{du}{dx}dx$ to leave just a *du*. This does not really make any sense as $\frac{du}{dx}$ is **not a fraction**, but is a good way to remember the substitution rule.

Example

Consider the integral

$$\int_0^2 x\cos(x^2+1)\,dx$$

By using the substitution $u = x^2 + 1$, we obtain $du = 2x\,dx$ and

$$\begin{aligned}\int_0^2 x\cos(x^2+1)\,dx &= \frac{1}{2}\int_0^2 \cos(x^2+1)2x\,dx \\ &= \frac{1}{2}\int_1^5 \cos(u)\,du \\ &= \frac{1}{2}(\sin(5)-\sin(1)).\end{aligned}$$

Note how the lower limit $x = 0$ was transformed into $u = 0^2 + 1 = 1$ and the upper limit $x = 2$ into $u = 2^2 + 1 = 5$.

Integration by Parts

Integration by parts for indefinite integrals

Suppose f(x) and g(x) are differentiable and their derivatives are continuous. Then

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

If we write u=f(x) and v=g(x) then using the Leibnitz notation du=f'(x)dx and dv=g'(x)dx and the integration by parts rule becomes

$$\int udv = uv - \int vdu.$$

Integration by parts for definite integrals

$$\int_a^b f(x)g'(x)dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$$
$$= f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

This can also be expressed in Leibniz notation.

$$\int_a^b udv = \Big[uv\Big]_a^b - \int_a^b vdu.$$

Example Find

$$\int x\cos(x)\,dx$$

Here we let:

$u = x$, so that $du = dx$,
$dv = \cos(x)dx$, so that $v = \sin(x)$.

Then:

$$\begin{aligned}\int x\cos(x)\,dx &= \int u\,dv\\ &= uv - \int v\,du\\ \int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx\\ \int x\cos(x)\,dx &= x\sin(x) + \cos(x) + C\end{aligned}$$

where C is an arbitrary constant of integration.

Example

$$\int x^2 e^x \, dx$$

In this example we will have to use integration by parts twice.

Here we let

$u = x^2$, so that $du = 2x dx$,
$dv = e^x dx$, so that $v = e^x$.

Then:

$$\int x^2 e^x \, dx = \int u \, dv$$
$$= uv - \int v \, du$$
$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

Now to calculate the last integral we use integration by parts again. Let

$u = x$, so that $du = dx$,
$dv = e^x dx$, so that $v = e^x$

and integrating by parts gives

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x.$$

So in the end $$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x) = x^2 e^x - 2x e^x + 2e^x = e^x(x^2 - 2x + 2).$$

Problems.

1. Integrate each of the following differential equations.

a.) $y' = x^2 - \sin x$

b.) $y' = 3t^2 + (\ln t/t)$

c.) $y' = (\cos x) \cdot e^{\sin x}$

d.) $y' = x\sin x^2$

e.) $y' = \sin x \cdot \cos x$

f.) $y' = x^2 \cdot \cos x^3 \cdot \sin x^3$

2. Use integration by parts to calculate each of the following integrals.

(a) $\int x \cdot e^{3x}\, dx$

(b) $\int x^2 \cos x\, dx$

(c) $\int x^2 e^{4x}\, dx$