# Limits

A limit looks at what happens to a function when the input approaches a certain value.
The general notation for a limit is as follows:

$$\lim_{x \to a} f(x)$$

This means, "The limit of f(x) as x approaches a".

Let's say that the function that we're interested in is $f(x) = x^2$, and that we're interested in its limit as x approaches 2. Using the above notation, we can write the limit as follows:

$$\lim_{x \to 2} x^2$$

Choose values near 2, compute f(x) for each, and see what happens as they get closer to 2.

| $x$ | 1.7 | 1.8 | 1.9 | 1.95 | 1.99 | 1.999 |
|---|---|---|---|---|---|---|
| $f(x) = x^2$ | 2.89 | 3.24 | 3.61 | 3.8025 | 3.9601 | 3.996001 |

Here we chose numbers smaller than 2, and approached 2 from below. We can also choose numbers larger than 2, and approach 2 from above:

| $x$ | 2.3 | 2.2 | 2.1 | 2.05 | 2.01 | 2.001 |
|---|---|---|---|---|---|---|
| $f(x) = x^2$ | 5.29 | 4.84 | 4.41 | 4.2025 | 4.0401 | 4.004001 |

As x grows closer and closer to 2, f(x) seems to get closer and closer to 4. So we can write:

$$\lim_{x \to 2} x^2 = 4.$$

Take another example: the function $f(x) = \frac{1}{x-2}$ as x approaches 2, or $\lim_{x\to 2} \frac{1}{x-2}$

Approaching from below:

| $x$ | 1.7 | 1.8 | 1.9 | 1.95 | 1.99 | 1.999 |
|---|---|---|---|---|---|---|
| $f(x) = \frac{1}{x-2}$ | -3.333 | -5 | -10 | -20 | -100 | -1000 |

Approaching from above:

| $x$ | 2.3 | 2.2 | 2.1 | 2.05 | 2.01 | 2.001 |
|---|---|---|---|---|---|---|
| $f(x) = \frac{1}{x-2}$ | 3.333 | 5 | 10 | 20 | 100 | 1000 |

In this case, the function doesn't seem to be approaching any value as x approaches 2. In this case we would say that the limit doesn't exist.

Now, consider the function $f(x) = \frac{x^2(x-2)}{x-2}$

This function is the same as: $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ \text{undefined} & \text{if } x = 2 \end{cases}$

So, for $x \neq 2$, the function $f(x) = \frac{x^2(x-2)}{x-2}$ gives the same outputs as $f(x) = x^2$

In terms of algebra, we could simply say that we can cancel the term $(x - 2)$, and then we have the function $f(x) = x^2$. This, however, is not quite correct; the function that we have now is not really the same as the one we started with, because it is defined at $x = 2$, and our original function was, specifically, not defined at $x = 2$. In algebra we had no better way of dealing with this type of function. Now, however, in calculus, we can introduce a more correct way of looking at this type of function.

What we want is to be able to say that the function $f(x) = \frac{x^2(x-2)}{x-2}$ doesn't exist

at $x = 2$, but for values very close to $x = 2$, the function gives outputs very close to 4. It may not get there, but it gets really, really close. The only question that we have is: what do we mean by "close"?

## Informal definition of a limit

We suppose that a function f is defined for values of x near c (but we do not require that it be defined when x = c).

**Definition: (Informal definition of a limit)**
We call $L$ the **limit of $f(x)$ as $x$ approaches $c$** if $f(x)$ becomes close to $L$ when $x$ is close (but not equal) to $c$.

When this holds we write

$$\lim_{x \to c} f(x) = L$$

or

$$f(x) \to L \quad \text{as} \quad x \to c.$$

Here, we are not concerned with the value of f(x) when x = c (which may exist or may not). All we care about are the values of f(x) when x is close to c, on either the left or the right (i.e. less or greater).

# Limit rules

The constant rule: if f(x) = b (that is, f is constant for all x), then the limit as x approaches c must be equal to b.

$$\lim_{x\to c} b = b.$$

The identity rule: if f(x) = x (that is, f just gives back whatever number you put in) then the limit of f as x approaches c is equal to c.

$$\lim_{x\to c} x = c.$$

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ and *k* is a constant, then:

scalar product $$\lim_{x\to c} kf(x) = k\cdot\lim_{x\to c} f(x) = kL$$

sum $$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$$

difference $$\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$$

product $$\lim_{x\to c}[f(x)\cdot g(x)] = \lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x) = L\cdot M$$

quotient $$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}\text{ as long as } M \neq 0$$

power rule $$\lim_{x\to c} f(x)^n = \left(\lim_{x\to c} f(x)\right)^n \text{ for a positive integer } n.$$

These rules are called identities.

## Examples

### Example 1

Find the limit $\lim_{x\to 2} 4x^3$.

We need to simplify this problem.

From the identity rule, $\lim_{x\to 2} x = 2$.

From the power rule, $\lim_{x\to 2} x^3 = \left(\lim_{x\to 2} x\right)^3 = 2^3 = 8$

Lastly, by the scalar product rule, we get

$$\lim_{x\to 2} 4x^3 = 4 \lim_{x\to 2} x^3 = 4 \cdot 8 = 32.$$

### Example 2

Find the limit $\lim_{x\to 2}[4x^3 + 5x + 7]$.

Split up the equation into its components.

From above, $\lim_{x\to 2} 4x^3 = 32$. , $\lim_{x\to 2} 5x = 5 \cdot \lim_{x\to 2} x = 5 \cdot 2 = 10$ and $\lim_{x\to 2} 7 = 7$.

Adding these together gives:

$$\lim_{x\to 2} 4x^3 + 5x + 7 = \lim_{x\to 2} 4x^3 + \lim_{x\to 2} 5x + \lim_{x\to 2} 7 = 32 + 10 + 7 = 49$$

## Example 3

Find the limit, $\lim\limits_{x \to 2} \dfrac{4x^3 + 5x + 7}{(x-4)(x+10)}$.

From the previous example the limit of the numerator is $\lim\limits_{x \to 2} 4x^3 + 5x + 7 = 49$.
The limit of the denominator is

$$\lim_{x \to 2}(x-4)(x+10) = \lim_{x \to 2}(x-4) \cdot \lim_{x \to 2}(x+10) = (2-4) \cdot (2+10) = -24.$$

As the limit of the denominator is not equal to zero we can divide which gives

$$\lim_{x \to 2} \frac{4x^3 + 5x + 7}{(x-4)(x+10)} = -\frac{49}{24}.$$

## Example 4

Find the limit, $\lim_{x \to 4} \frac{x^4 - 16x + 7}{4x - 5}$.

We apply the same process here as we did in the previous set of examples;

$$\lim_{x \to 4} \frac{x^4 - 16x + 7}{4x - 5} = \frac{\lim_{x \to 4}(x^4 - 16x + 7)}{\lim_{x \to 4}(4x - 5)} = \frac{\lim_{x \to 4}(x^4) - \lim_{x \to 4}(16x) + \lim_{x \to 4}(7)}{\lim_{x \to 4}(4x) - \lim_{x \to 4} 5}.$$

We can evaluate each of these;

$\lim_{x \to 4}(x^4) = 256$, $\lim_{x \to 4}(16x) = 64$, $\lim_{x \to 4}(7) = 7$, $\lim_{x \to 4}(4x) = 16$ and

$\lim_{x \to 4}(5) = 5$. Thus, the answer is $\frac{199}{11}$.

## Example 5

Find the limit $$\lim_{x \to 0} \frac{1 - \cos x}{x}.$$

To evaluate this seemingly complex limit, we will need to recall some sine and cosine identities. We will also have to use two new facts.

1. If f(x) is a trigonometric function (i.e. one containing sine, cosine, tangent, cotangent, secant or cosecant) and is defined at a, then: $$\lim_{x \to a} f(x) = f(a).$$

2. $$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

To start, recognize that 1- cosx can be multiplied by 1 + cosx to obtain (1 - cos$^2$x) which, by our trig identities, is sin$^2$x. So, multiply the top and bottom by 1 + cosx.

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \left( \frac{1 - \cos x}{x} \cdot \frac{1}{1} \right) \\
&= \lim_{x \to 0} \left( \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \right) \\
&= \lim_{x \to 0} \frac{(1 - \cos x) \cdot 1 + (1 - \cos x) \cdot \cos x}{x \cdot (1 + \cos x)} \\
&= \lim_{x \to 0} \frac{1 - \cos x + \cos x - \cos^2 x}{x \cdot (1 + \cos x)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \to 0} \frac{1 - \cos^2 x}{x \cdot (1 + \cos x)} \\
&= \lim_{x \to 0} \frac{\sin^2 x}{x \cdot (1 + \cos x)} \\
&= \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \right).
\end{aligned}$$

Next, break this up into $\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x}$ by the product rule.

As mentioned above, $\lim_{x\to 0}\frac{\sin x}{x}=1$

Next $\lim_{x\to 0}\frac{\sin x}{1+\cos x}=\frac{\lim_{x\to 0}\sin x}{\lim_{x\to 0}(1+\cos x)}=\frac{0}{1+\cos 0}=0$

$\therefore$ $\lim_{x\to 0}\frac{1-\cos x}{x}$ = 1 × 0 = 0.

## Limits of Polynomials and Rational functions

We can find the limit at c of any polynomial or rational function, as long as that rational function is defined at c (so we are not dividing by zero).

If $f$ is a polynomial or rational function that is defined at $c$ then

$$\lim_{x \to c} f(x) = f(c)$$

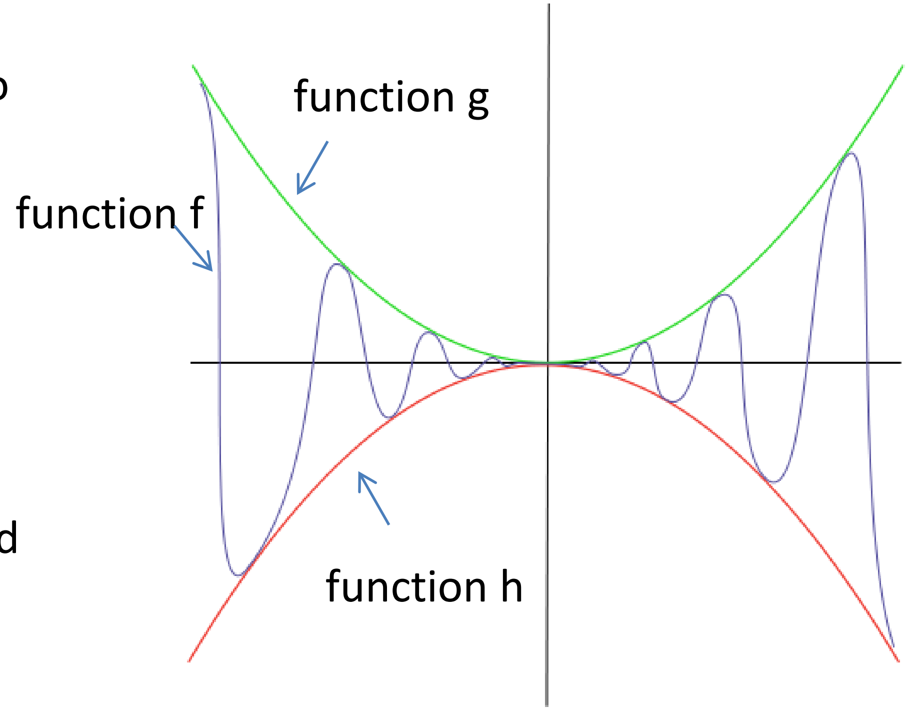
We already learned this for trigonometric functions, so we see that it is easy to find limits of polynomial, rational or trigonometric functions wherever they are defined. In fact, this is true even for combinations of these functions; thus, for example,

$$\lim_{x \to 1} (\sin x^2 + 4\cos^3(3x - 1)) = \sin 1^2 + 4\cos^3(3(1) - 1)$$

## The Squeeze Theorem

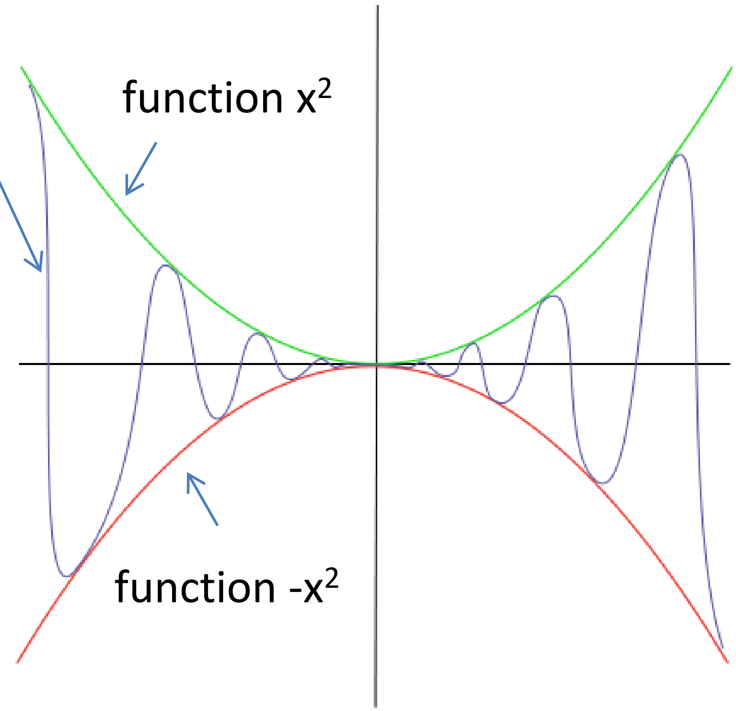
The Squeeze Theorem is typically used to find the limit of a function by comparison with two other functions whose limits are known.

It is called the Squeeze Theorem because it refers to a function f whose values are squeezed between the values of two other functions g and h, both of which have the same limit L. If the value of f is trapped between the values of the two functions f and g, the values of f must also approach L.




Suppose that g(x) ≤ f(x) ≤ h(x) holds for all x in some open interval (i.e. a set of numbers) containing a, except possibly at x = a itself.
Suppose also that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ . Then $\lim_{x\to a} f(x) = L$

## Example




The limit $\lim_{x \to 0} x^2 \sin(\frac{1}{x})$

cannot be ascertained through the limit law

$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x),$$

because $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

However, by the definition of the sine function*,

$$-1 \leq \sin(\tfrac{1}{x}) \leq 1.$$

It follows that $-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$

Since $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$ , by the squeeze theorem, $\lim_{x \to 0} x^2 \sin(\frac{1}{x})$ must also be 0.

*The sine of anything is in the interval [-1,1].

# Differentiation

Differentiation is a method that allows us to find a function that relates the rate of change of one variable with respect to another variable.

The fundamental concept of calculus (differentiation and integration) is the theory of limits of functions. A function is a defined relationship between two or more variables. One variable, called the dependent variable, approaches a limit as another variable, called the independent variable, approaches a number or becomes infinite.

For instance, the amount of postage required to mail a package is related to its weight. Here, the *weight* of the package is considered the *independent variable*, and the *amount of postage* is considered the *dependent variable*.

## Differential Calculus

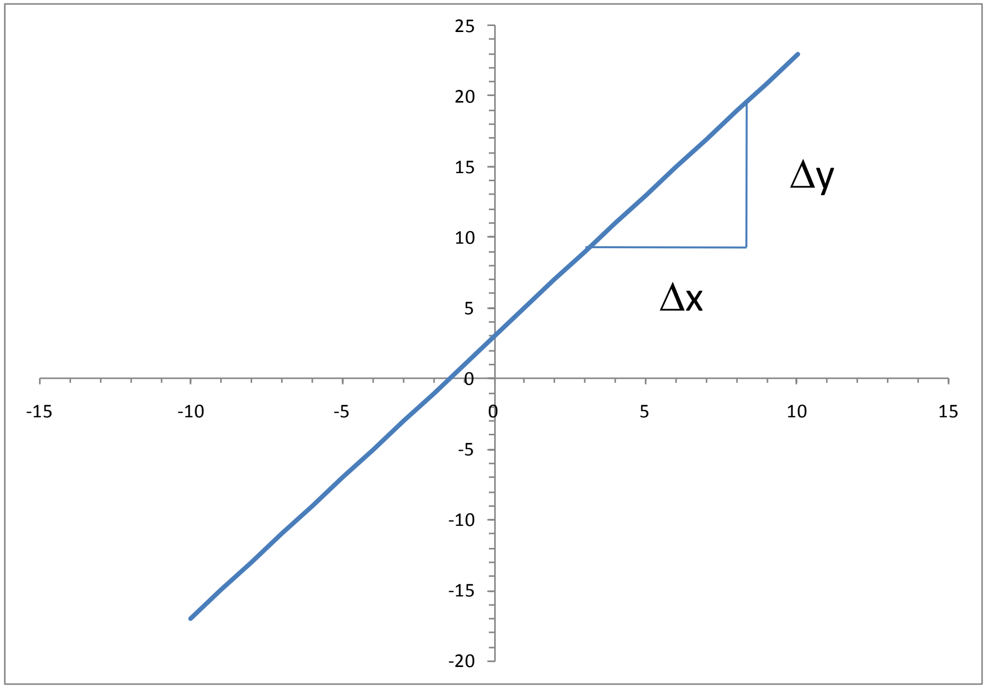
The independent variable is often called x.
The dependent variable is often called y.
If y is a function of x, this is written as y = f(x)

If the variable x changes by a particular amount h, the variable y will also change by a predictable amount k. The ratio of the two amounts of change k/h is called a *difference quotient*. If the rate of change of y differs as x changes, this quotient indicates the *average* rate of change of y = f(x) over the amount h.

If the ratio k/h has a limit as h approaches 0, this limit is called the *derivative* of y. The derivative of y may be interpreted as the slope of the curve graphed by the equation y = f(x), measured at a particular point. It may also be interpreted as the instantaneous rate of change of y. The process of finding a derivative is called *differentiation*.

# The Definition of Slope

The slope of a line, also called the *gradient* of the line, is a measure of its inclination. A line that is horizontal has slope 0, a line from the bottom left to the top right has a positive slope, a line from the top left to the bottom right has a negative slope.




Gradient can be defined by measuring how much the line climbs for a given "step" horizontally. We denote a step in a quantity using a delta (Δ) symbol. Thus, a step in x is written as Δx. We can therefore write this definition of gradient as:

$$\boxed{\text{Gradient} = \frac{\Delta y}{\Delta x}}$$

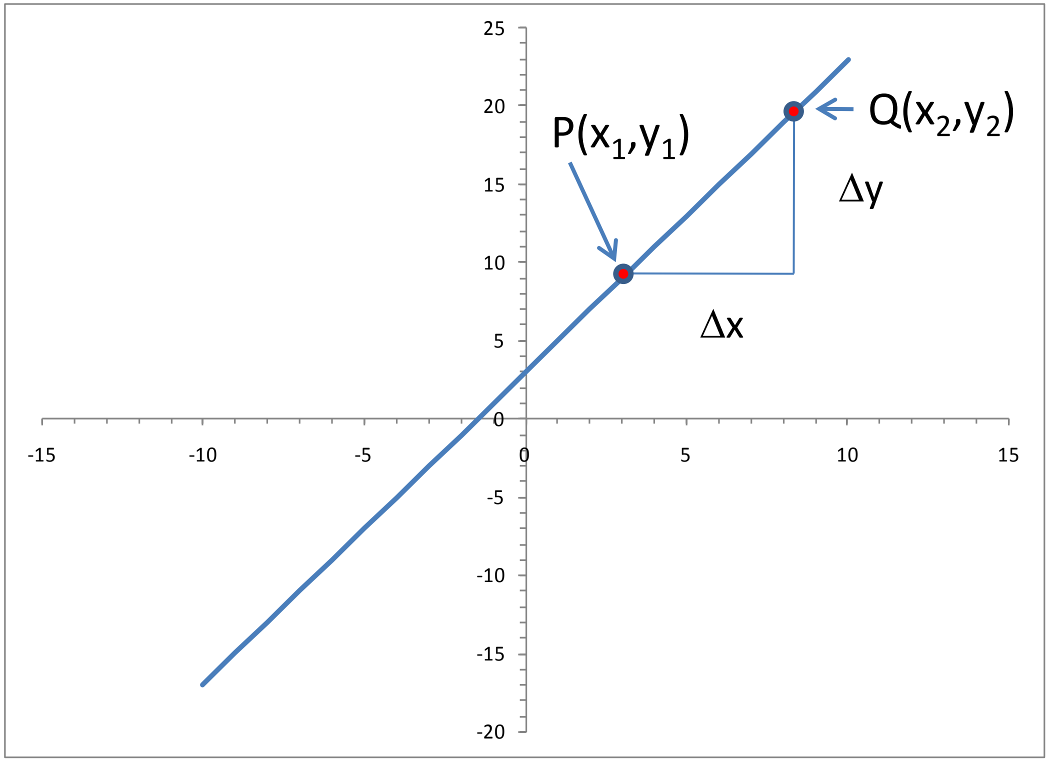
If we have two points on a line, $P(x_1,y_1)$ and $Q(x_2,y_2)$, the step in x from P to Q is given by:

$\Delta x = x_2 - x_1$

Likewise, the step in y from P to Q is given by:

$\Delta y = y_2 - y_1$

This leads to the very important result below.

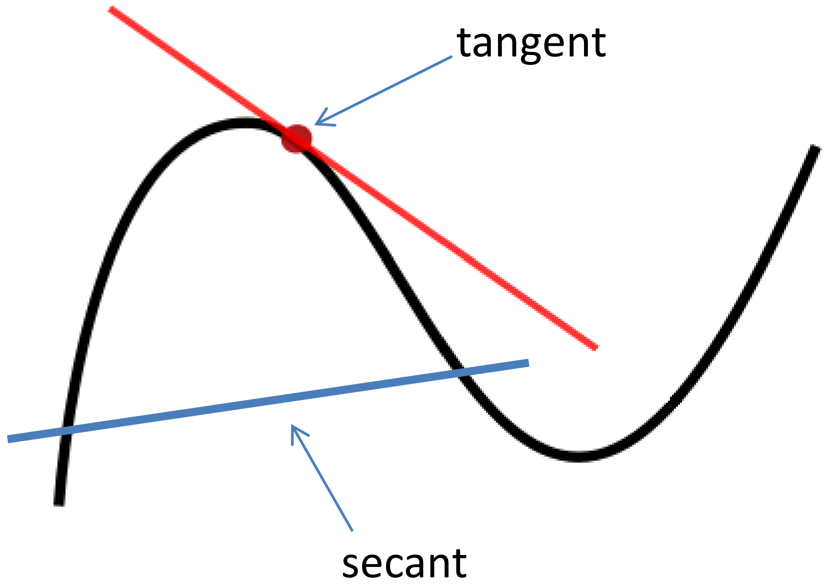



The definition of slope, $m$, between two points $(x_1,y_1)$ and $(x_2,y_2)$ on a line is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

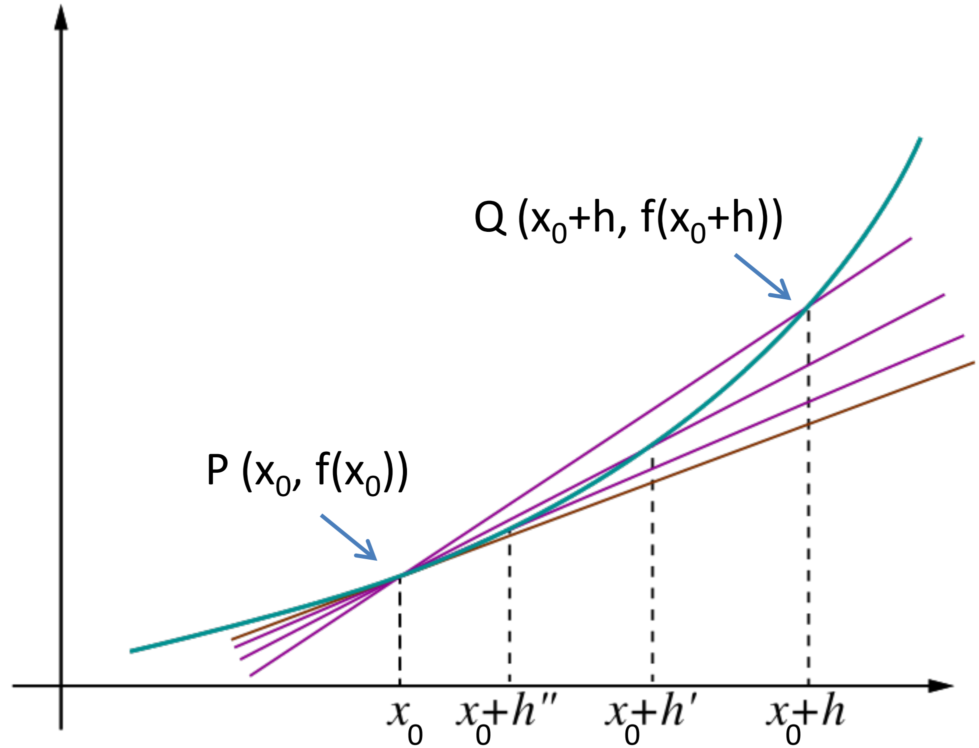
## On a Function

Most functions we are interested in are not straight lines. We cannot define a gradient of a curved function in the same way as we can for a line. In order for us to find the gradient of a function at a point, we have to draw a *tangent*. A tangent is a line which just touches a curve at a point, such that the angle between them at that point is zero.




A *secant* is a line drawn through two points on a curve. We can construct a definition of the tangent as the limit of a secant of the curve drawn as the separation between the points tends to zero. Consider the diagram below.

Draw a secant through the points P ($x_0$, f($x_0$)) and Q ($x_0$+h, f($x_0$+h)). As the distance h tends to zero, the secant line becomes the tangent at the point P.




We can find the slope, m, of the secant from before:

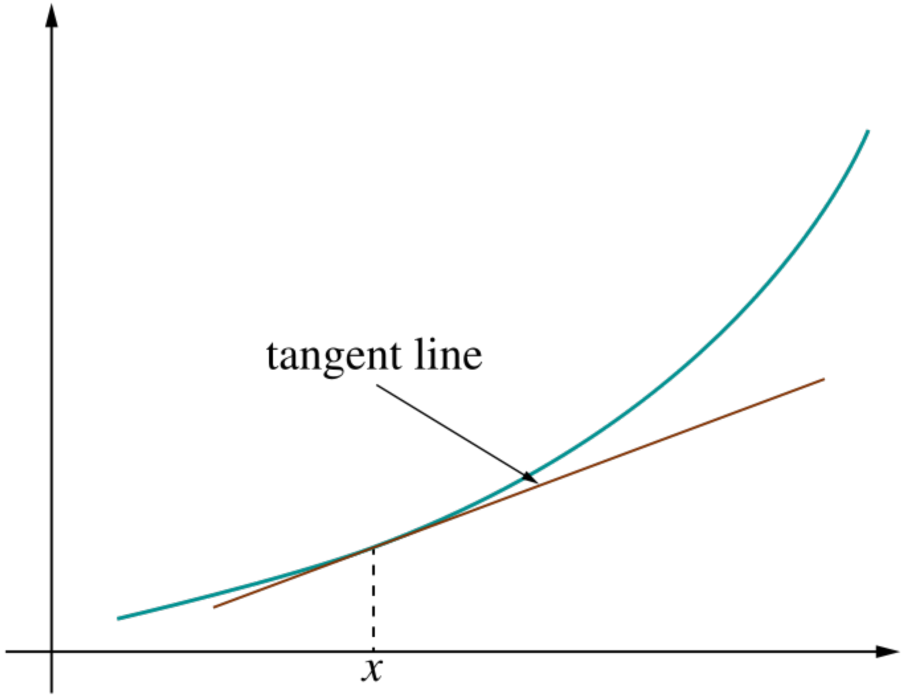
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting in the points on the line, $$m = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0}.$$

This simplifies to $$m = \frac{f(x_0+h) - f(x_0)}{h}.$$

This expression is called the *difference quotient*. Note that h can be positive or negative.

Now, to find the slope of the tangent, $m_0$ we let h aproach zero. We cannot simply set it to zero as this would imply division of zero  by zero which would yield an undefined result. Instead we must find the limit of the above expression as h *tends* to zero:




$$\boxed{m_0 = \lim_{h \to 0} \left[ \frac{f(x_0 + h) - f(x_0)}{h} \right]}$$

## The Slope at a Point

Consider a car which travels a distance x (e.g. 600 km) in a time t (e.g. 10 hours).

The *average* velocity of the car during this time is given by: $\mathrm{v} = \dfrac{\Delta \mathrm{x}}{\Delta \mathrm{t}} = \dfrac{600\,\mathrm{km}}{10\,\mathrm{h}} = 60\,\mathrm{km/h}$

If we want to know the velocity at a particular time t (*instantaneous* velocity), we need to look at the distance travelled *as the change in time approaches 0*.

This is written as: $\lim_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t} = \dfrac{dx}{dt}$

$\Delta$ and *d* both indicate a difference, but *d* indicates an infinitesimal (very small) difference.

The letter s is often used to denote distance, which gives $\dfrac{ds}{dt}$.

If a function f(x) is plotted on an (x,y) Cartesian coordinate system, differentiation will yield a function which describes the rate of change of y with respect to x.

## The Definition of the Derivative

For a function y = f(x), its derivative can be expressed as:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \left[ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

The derivative f'(x) can be expressed in several ways:

$\frac{dy}{dx}$ Leibniz 's notation: You may think of this as "rate of change in y with respect to x". You may also think of it as "infinitesimal value of y divided by infinitesimal value of x".

$\frac{dy}{dx}, \quad \frac{df}{dx}(x), \quad \text{or} \quad \frac{d}{dx}f(x),$ are all equivalent.

$f'(x)$ Lagrange's notation

$\dot{y}$ Newton's notation: This is used to represent derivatives of time i.e. for functions of the form y = f(t)

$D_x y$ or $D_x f(x)$ Euler's notation

## Examples

For the function f(x) = x/2

$$f(x) = \frac{x}{2}$$

$$f'(x) = \lim_{\Delta x \to 0} \left( \frac{\frac{x+\Delta x}{2} - \frac{x}{2}}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left( \frac{1}{2} \right) = ½$$

For the function f(x) = $x^2$

$$f(x) = x^2$$

$$f'(x) = \lim_{\Delta x \to 0} \left[ \frac{(x + \Delta x)^2 - x^2}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left( \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left( \frac{2x\Delta x + \Delta x^2}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x$$

## Differentiation of a polynomial

Find the derivative of $y = f(x) = x^3 - 3x^2 + 4x + 2$

First, compute f(x + Δx) - f(x). So we replace x with x + Δx in the above equation, then we subtract f(x) from it:

$$f(x+\Delta x) - f(x) = (x+\Delta x)^3 - 3(x+\Delta x)^2 + 4(x+\Delta x) + 2 - (x^3 - 3x^2 + 4x + 2)$$

$$= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 3(x^2 + 2x\Delta x + \Delta x^2) + 4x + 4\Delta x + 2 - (x^3 - 3x^2 + 4x + 2)$$

$$= x^3 - x^3 - 3x^2 + 3x^2 + 4x - 4x + 2 - 2 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 6x\Delta x - 3\Delta x^2 + 4\Delta x$$

$$= 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 6x\Delta x - 3\Delta x^2 + 4\Delta x$$

$$= \Delta x(3x^2 + 3x\Delta x + \Delta x^2 - 6x - 3\Delta x + 4)$$

Substituting the last line into the expression for a derivative gives:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + \Delta x^2 - 6x - 3\Delta x + 4)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + \Delta x^2 - 6x - 3\Delta x + 4) = 3x^2 - 6x + 4$$

Note: division by Δx on the right side term in the second line must be performed before we let $\Delta x \to 0$.

# Differentiation rules

Derivative of a Constant Function: for a fixed (constant) real number *c* $\frac{d}{dx}[c] = 0$

**Example**

$$\frac{d}{dx}[3] = 0$$

Derivative of a Linear Function: for any fixed real numbers m and c $\frac{d}{dx}[mx + c] = m$

Example $\frac{\mathrm{d}}{\mathrm{dx}}[4\mathrm{x}+5]=4$

The Constant Rule: for any fixed real number c $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$

Example

We already know that

$$\frac{d}{dx}\left[x^2\right] = 2x.$$

Suppose we want to find the derivative of $3x^2$

$$\frac{d}{dx}\left[3x^2\right] = 3\frac{d}{dx}\left[x^2\right]$$
$$= 3 \times 2x$$
$$= 6x$$

Addition and Subtraction Rules: $$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example

$$\begin{aligned}\frac{d}{dx}\left[3x^2 + 5x\right] &= \frac{d}{dx}\left[3x^2 + 5x\right] \\ &= \frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}[5x] \\ &= 6x + \frac{d}{dx}[5x] \\ &= 6x + 5\end{aligned}$$

The power rule: $$\frac{d}{dx}[x^n] = nx^{n-1}, x \neq 0$$

Example $$\frac{d}{dx}\left[x^2\right] = 2x^{2-1} = 2x^1 = 2x$$

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left[kx^n\right] = knx^{n-1}$$

Example $$\frac{d}{dx}\left[3x^4\right] = 3 \cdot 4x^{4-1} = 12x^3$$

You can now find the derivative of any polynomial you come across.

Example: Differentiate y = $6x^5 + 3x^2 + 3x + 1$

$$\frac{d}{dx}\left[6x^5 + 3x^2 + 3x + 1\right]$$

The first thing we can do is to use the addition rule to split the equation up into terms:

$$\frac{d}{dx}\left[6x^5\right] + \frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}\left[3x\right] + \frac{d}{dx}\left[1\right].$$

We can immediately use the linear and constant rules to get rid of some terms:

$$\frac{d}{dx}\left[6x^5\right] + \frac{d}{dx}\left[3x^2\right] + 3 + 0.$$

Now you may use the constant multiplier rule to move the constants outside the derivatives:

$$6\frac{d}{dx}\left[x^5\right] + 3\frac{d}{dx}\left[x^2\right] + 3.$$

Then use the power rule to work with the individual monomials:

$$6\left(5x^4\right) + 3\left(2x\right) + 3.$$

Which gives us the final answer:

$$30x^4 + 6x + 3.$$

Exercises

- Find the derivatives of the following equations:

$f(x) = 42$
$f(x) = 6x + 10$
$f(x) = 2x^2 + 12x + 3$

## Higher-Order Derivatives

When we differentiate a function, we get another function.

y = f(x) $\rightarrow$ y' = f'(x) *first derivative*

The second derivative of the function tells us the rate of change of the first derivative.

We can differentiate this function again.

y' = f'(x) $\rightarrow$ y'' = f''(x) *second derivative*

$\frac{d^2y}{dx^2}$ in Leibniz's notation $\ddot{y}$ in Newton's notation

$D_x^2 y \text{ or } D_x^2 f(x)$ in Euler's notation

## Extrema and Points of Inflexion

Maximum: a point where a function reaches its highest value

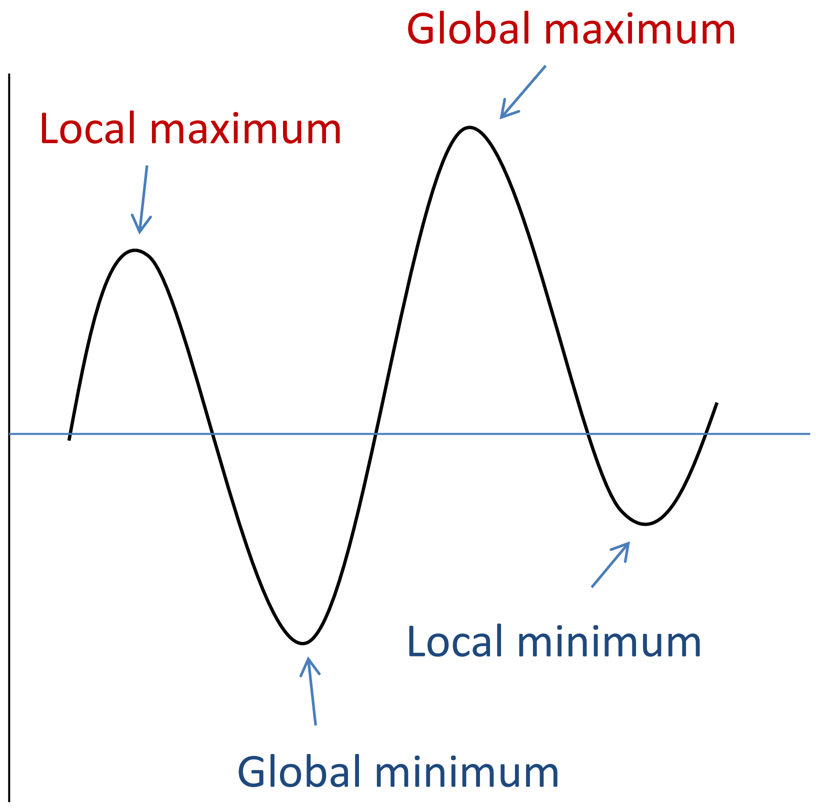
Minimum: a point where a function reaches its lowest value

Extremum: maximum or minimum value

Global maximum: a point that takes the largest value on the entire range of the function

Global minimum: a point that takes the smallest value on the entire range of the function

local maxima / minima are the largest / smallest values of the function in the immediate vicinity.



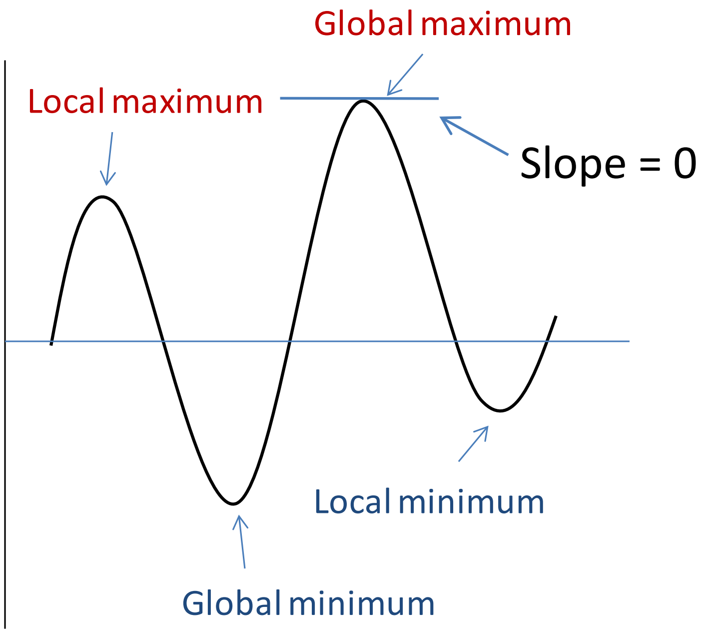

N.B. -um is singular, -ma is plural

At any extremum, the slope of the graph is necessarily zero, as the graph must stop rising or falling at an extremum, and begin to fall or rise.

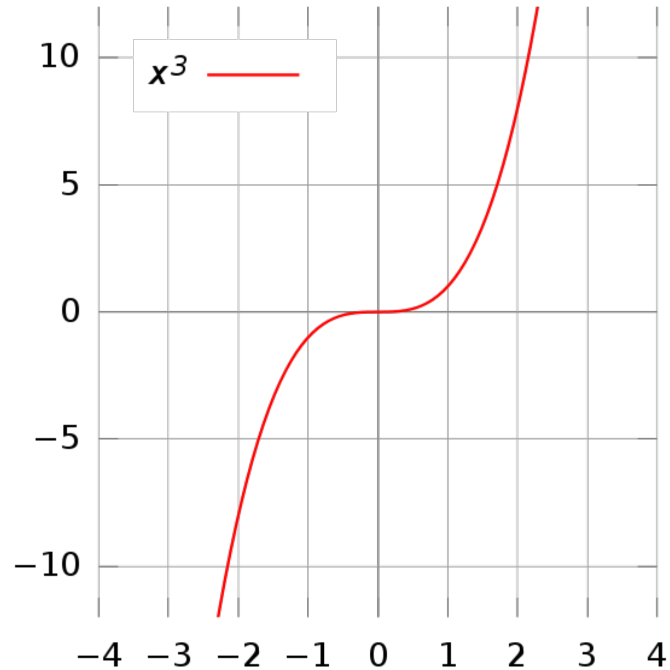
Extrema are also commonly called stationary points or turning points.

The first derivative of a function is equal to zero at extrema i.e. f'(x) = 0.

If the graph has one or more of these stationary points, these may be found by setting the first derivative equal to zero and finding the roots of the resulting equation.

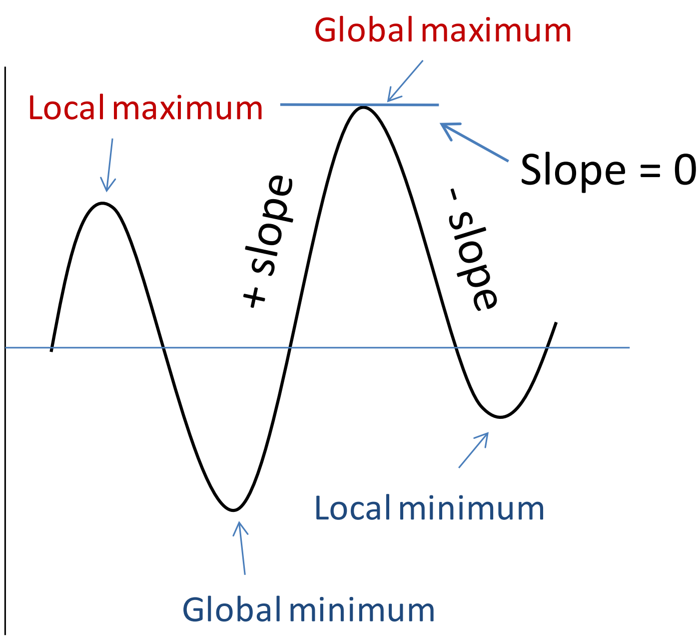


However, a slope of zero does not guarantee a maximum or minimum: there is a third class of stationary point called a point of inflexion. Consider the function $f(x) = x^3$

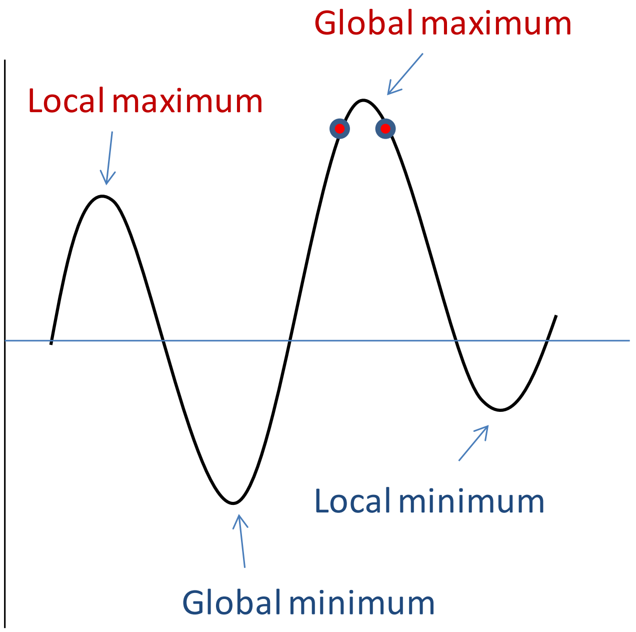



The derivative is $f'(x) = 3x^2$

The slope at x=0 is 0. This is a stationary point, but it isn't a maximum or minimum. Looking at the graph of the function, x=0 is just a spot at which the function flattens out. True extrema require the a sign change in the first derivative. This makes sense - you have to rise (positive slope) to and fall (negative slope) from a maximum. In between rising and falling, on a smooth curve, there will be a point of zero slope - the maximum. A minimum would exhibit similar properties, just in reverse.



This leads to a simple method to classify a stationary point - plug x values slightly left and right of the point into the derivative of the function. If the results have opposite signs then it is a true maximum/minimum. You can also use these slopes to figure out if it is a maximum or a minimum: the left slope will be positive for a maximum and negative for a minimum.



## The Extremum Test

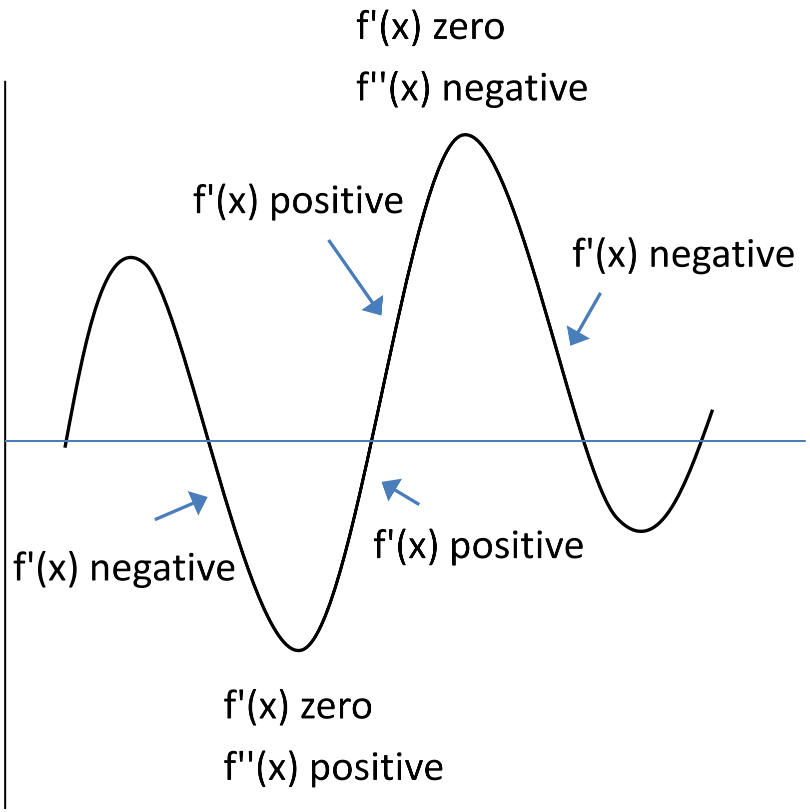
The sign of the first derivative must change for a stationary point to be a true extremum.

The *second* derivative of the function tells us the rate of change of the first derivative.

If the second derivative is positive at the stationary point, then the gradient is increasing (changing from negative to positive in this case).

The fact that it is a stationary point in the first place means that this can only be a minimum.

If the second derivative is negative at that point, then it is a maximum.



If the second derivative is zero, we have a problem. It could be a point of inflexion, or it could still be an extremum.

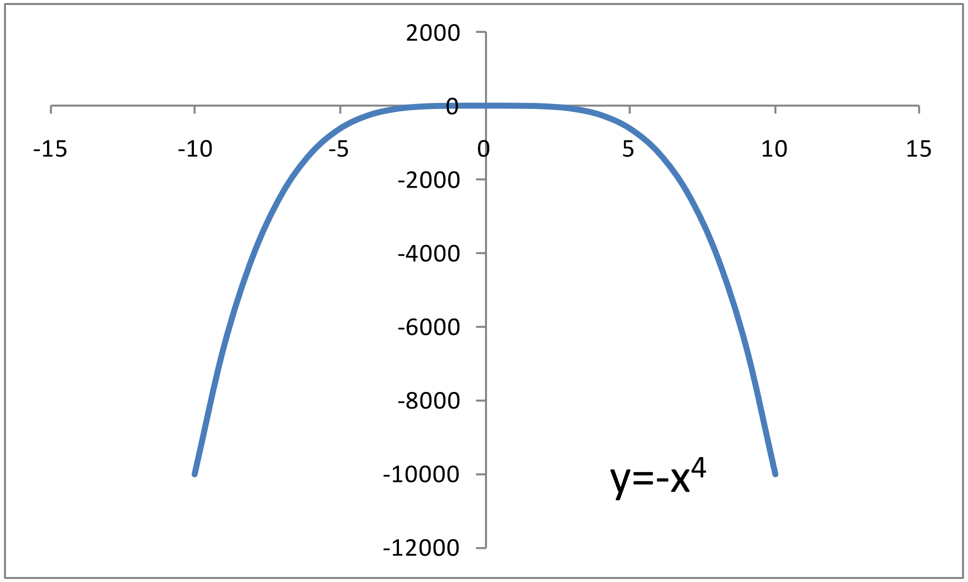
What we must do is continue to differentiate until we get, at the (n+1)th derivative, a non-zero result at the stationary point:

$$f'(x) = 0,\ f''(x) = 0,\ \ldots, f^{(n)}(x) = 0,\ f^{(n+1)}(x) \neq 0$$

- If n is odd, then the stationary point is a true extremum.
- If the (n+1)th derivative is positive, it is a minimum; if the (n+1)th derivative is negative, it is a maximum.
- If n is even, then the stationary point is a point of inflexion.

Example

$y = -x^4$ has a stationary point at $x = 0$
$y' = -4x^3$, which also $= 0$ at $x = 0$
$y'' = -12x^2$, which also $= 0$ at $x = 0$
$y''' = -24x$, which also $= 0$ at $x = 0$
$y^{(4)} = -24$




Therefore, (n+1) is 4, so n is 3. This is odd, so the stationary point is a true extremum. The fourth derivative is negative, so we have a maximum.

## Chain Rule

We know how to differentiate regular polynomial functions. For example:

$$f(x) = (x^2 + 5)^2$$
$$f(x) = x^4 + 10x^2 + 25$$
$$f'(x) = 4x^3 + 20x$$

We can also differentiate the function $f(x) = (x^2 + 5)^2$ by using the *chain rule*.

The function can be written as $f(x) = u^2$, where $u = m(x) = (x^2 + 5)$.

Now, let $f(x) = g(u) = g(m(x))$. Then $f'(x) = g'(u)m'(x) = g'(m(x))m'(x)$

The chain rule states that if we have a function of the form $y(u(x))$ (i.e. y can be written as a function of u and u can be written as a function of x) then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

**Chain Rule**

If a function F(x) is composed to two differentiable functions g(x) and m(x), so that F(x)=g(m(x)), then F(x) is differentiable and,

$$F'(x) = g'(m(x))m'(x)$$

Going back to $f(x) = (x^2 + 5)^2$: $y = (x^2 + 5)^2 = u^2$ (where $u = x^2 + 5$)

$$\frac{dy}{du} = 2u \text{ and } \frac{du}{dx} = 2x \quad \therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 2x = 2(x^2 + 5)(2x) = 4x^3 + 20x$$

Consider another example $\frac{d}{dx}\sqrt{1 + x^2}$ Let $y = \sqrt{u}$ where $u = 1 + x^2$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \text{ and } \frac{du}{dx} = 2x \quad \therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{1 + x^2}} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}$$

## Product and Quotient Rules

When we wish to differentiate a more complicated expression such as: $h(x) = (x^3 + 3x + 1)(x^2 + 2)$

We can expand it and get a polynomial, and then differentiate that polynomial. This method becomes very complicated and it's easy to make mistakes.

We can split the function h(x) into two functions $f(x) = (x^3 + 3x + 1)$ and $g(x) = (x^2 + 2)$ and then find the derivative of h(x) using the derivatives of f(x) and g(x).

**Derivatives of products (Product rule)**

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$f'(x) = 3x^2 + 6x$

$g'(x) = 2x$

$$\begin{aligned} h'(x) &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \\ &= (x^3 + 3x^2 + 1)(2x) + (x^2 + 2)(3x^2 + 6x) \\ &= 5x^4 + 12x^3 + 6x^2 + 14x \end{aligned}$$

# Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$$

## Examples

The derivative of $(4x - 2) / (x^2 + 1)$ is:

$$\begin{aligned}\frac{d}{dx}\left[\frac{(4x-2)}{x^2+1}\right] &= \frac{(x^2+1)(4) - (4x-2)(2x)}{(x^2+1)^2}\\ &= \frac{(4x^2+4) - (8x^2-4x)}{(x^2+1)^2}\\ &= \frac{-4x^2+4x+4}{(x^2+1)^2}\end{aligned}$$

## Exponential, logarithmic, and trigonometric functions

### Exponentials

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = \ln(a)\, a^x$$

### Logarithms

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$$

### Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{dx}} sin(\mathrm{x}) = cos(\mathrm{x})$$

$$\frac{\mathrm{d}}{\mathrm{dx}} cos(\mathrm{x}) = -sin(\mathrm{x})$$

$$\frac{\mathrm{d}}{\mathrm{dx}} tan(\mathrm{x}) = sec^2(\mathrm{x})$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

### Inverse Trigonometric Functions

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Exercises

1. Find the derivatives of the following functions: a.) $y = 2x^6 - 3x^2$

b.) $y = \frac{1}{x}$

c.) $y = \sqrt{4x}$

2. Find $y'$, given $y = \frac{x+1}{x^2+1}$.

Find $y'$, given $(a)$ $y = 1/x^3$,

$(b)$ $y = \frac{2x}{x-3}$,

$(c)$ $y = \frac{x+5}{x^2-1}$,