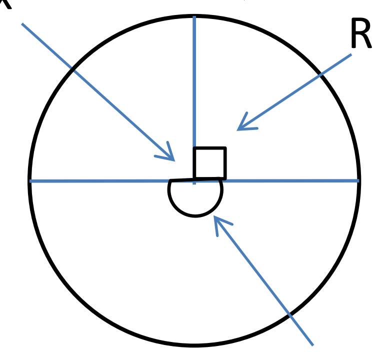
Fundamentals of Trigonometry

Trigonometry is the branch of mathematics that is concerned with the relationships between the sides and the angles of triangles.

An *angle is an angular unit of measure with a vertex (the point at which the sides of an angle intersect)* at the center of a circle, and with sides that subtend (cut off) part of the circumference.



If the subtended arc is equal to one−fourth of the total circumference, the angular unit is a *right angle. If the arc equals half the circumference, the unit is a straight angle (an angle of 180°). If the* arc equals 1/360 of the circumference, the angular unit is one *degree.*

Acute angle: $0° < \theta < 90°$
Obtuse angle: $90° < \theta < 180°$

Each degree is subdivided into 60 equal parts called minutes, and each minute is subdivided into 60 equal parts called seconds. The symbol for degree is °; for minutes ', and for seconds ''. In trigonometry, it is customary to denote angles with the Greek letters θ (theta) or φ (phi).

Another unit of angular measure is the *radian*.

The radian is a circular angle subtended by an arc equal in length to the radius of the circle whose radius is r units in length. The circumference of a circle is $2\pi r$ units; therefore, there are 2π radians in 360°. Also, π radians corresponds to 180°.

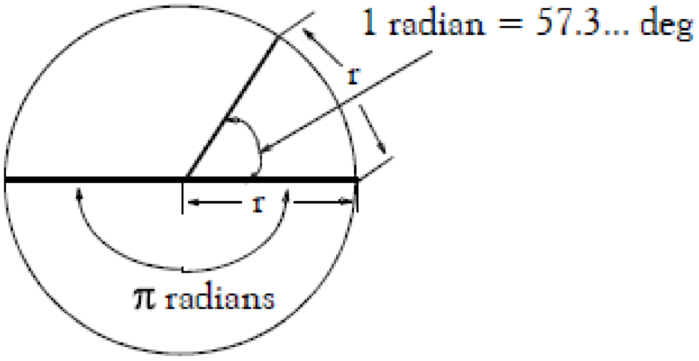


Figure 4.1. Definition of the radian

Trigonometric Functions

Sine of Angle A = sinA = $\frac{a}{c}$ $= \frac{\text{Opposite}}{\text{Hypotenuse}}$

Cosine of Angle A = cosA = $\frac{b}{c}$ $= \frac{\text{Adjacent}}{\text{Hypotenuse}}$

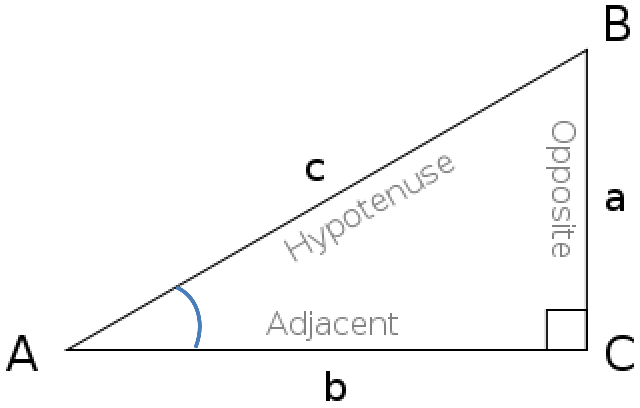
Tangent of Angle A = tanA = $\frac{a}{b}$ $= \frac{\text{Opposite}}{\text{Adjacent}}$

Cotangent of Angle A = cotA = $\frac{b}{a}$ $= \frac{1}{\tan A}$

Secant of Angle A = secA = $\frac{c}{b}$ $= \frac{1}{\cos A}$

Cosecant of Angle A = cscA = $\frac{c}{a}$ $= \frac{1}{\sin A}$

$$\tan A = \frac{\sin A}{\cos A}$$



Mnemonics

SOH-CAH-TOA

Sine = Opposite ÷ Hypotenuse
Cosine = Adjacent ÷ Hypotenuse
Tangent = Opposite ÷ Adjacent

"Some Old Horses Chew Apples Happily Throughout Old Age"

Trigonometric Functions of an Any Angle

Quadrant: any of the four areas into which a plane is divided by the reference set of axes in a Cartesian (x,y) coordinate system

To generate an angle in the 1st quadrant:

1. Take a line OP_0 which lies along the positive x-axis.
2. Rotate it counterclockwise about its origin 0 onto ray OP to form a positive angle θ_1 . This angle is greater than 0 but less than 90°.

Denoting the distance OP as r we obtain:

$$\sin\theta_1 = \frac{y}{r} \qquad \cot\theta_1 = \frac{x}{y}$$

$$\cos\theta_1 = \frac{x}{r} \qquad \sec\theta_1 = \frac{r}{x}$$

$$\tan\theta_1 = \frac{y}{x} \qquad \csc\theta_1 = \frac{r}{y}$$

For quadrant I, both x and y are positive.

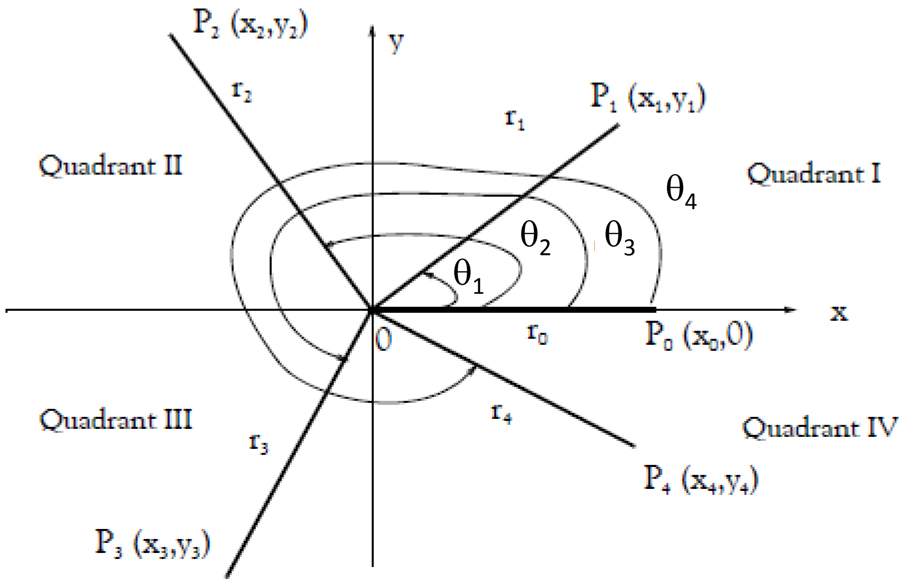
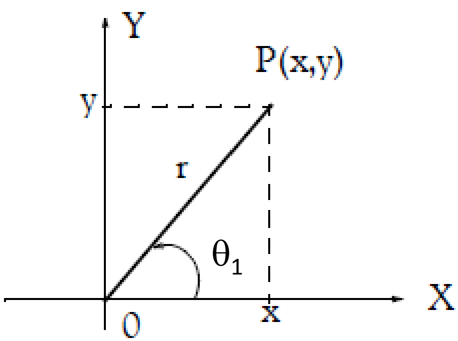


Figure 4.3. Quadrants in Cartesian Coordinates



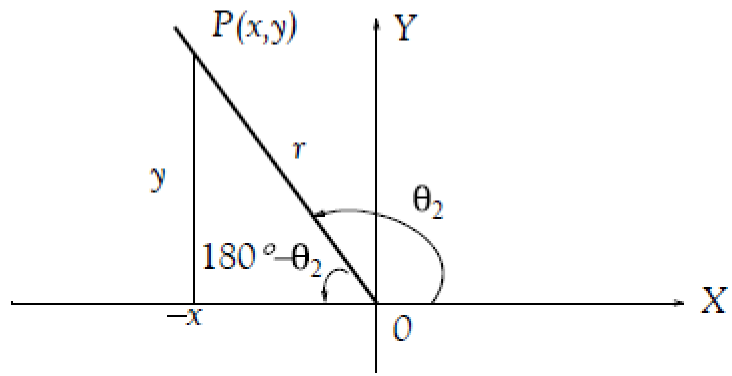
Definition of trigonometric functions of angles in first quadrant

If we look at the 2nd quadrant:

$90° < \theta_2 < 180°$

x is negative, y is positive

Using trigonometric reduction formulas, we obtain:



Definition of trigonometric functions of angles in second quadrant.

$$\sin(180° - \theta_2) = \sin\theta_2 = \frac{y}{r}$$

$$\cos(180° - \theta_2) = -\cos\theta_2 = \frac{-x}{r}$$

$$\tan(180° - \theta_2) = -\tan\theta_2 = \frac{y}{-x}$$

$$\cot(180° - \theta_2) = \frac{1}{\tan(180° - \theta_2)} = \frac{1}{-\tan\theta_2} = -\cot\theta_2 = \frac{-x}{y}$$

$$\sec(180° - \theta_2) = \frac{1}{\cos(180° - \theta_2)} = \frac{1}{-\cos\theta_2} = -\sec\theta_2 = \frac{r}{-x}$$

$$\csc(180° - \theta_2) = \frac{1}{\sin(180° - \theta_2)} = \frac{1}{\sin\theta_2} = \csc\theta_2 = \frac{r}{y}$$

The cosine, tangent, cotangent and secant functions are negative in the second quadrant.

TABLE 4.1 *Signs of the trigonometric functions in different quadrants.*

Quadrant	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
I	+	+	+	+	+	+
II	+	–	–	–	–	+
III	–	–	+	+	–	–
IV	–	+	–	–	+	–

r is the hypotenuse of the right triangle formed by it and the line segments x and y. Therefore, x and y can never be greater in length than r. Accordingly, the sine and the cosine functions can never be greater than 1, that is, they can only vary from 0 to ±1 inclusive.

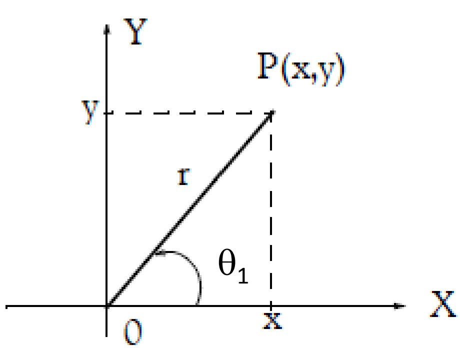


TABLE 4.2 *Limits of the trigonometric functions*

Quadrant	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
I	0 to +1	+1 to 0	0 to +∞	+∞ to 0	+1 to +∞	+∞ to +1
II	+1 to 0	0 to –1	–∞ to 0	0 to –∞	–∞ to –1	+1 to +∞
III	0 to –1	–1 to 0	0 to +∞	+∞ to 0	–1 to –∞	–∞ to –1
IV	–1 to 0	0 to +1	–∞ to 0	0 to –∞	+∞ to +1	–1 to –∞

Sines and cosines of common angles

$$\cos 0^\circ = \cos 360^\circ = \cos 2\pi = 1$$
$$\cos 30^\circ = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = 0.866$$
$$\cos 45^\circ = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = 0.707$$
$$\cos 60^\circ = \cos\frac{\pi}{3} = \frac{1}{2} = 0.5$$
$$\cos 90^\circ = \cos\frac{\pi}{2} = 0$$
$$\cos 120^\circ = \cos\frac{2\pi}{3} = \frac{-1}{2} = -0.5$$
$$\cos 150^\circ = \cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2} = -0.866$$
$$\cos 180^\circ = \cos\pi = -1$$
$$\cos 210^\circ = \cos\frac{7\pi}{6} = \frac{-\sqrt{3}}{2} = -0.866$$
$$\cos 225^\circ = \cos\frac{5\pi}{4} = \frac{-\sqrt{2}}{2} = -0.707$$
$$\cos 240^\circ = \cos\frac{4\pi}{3} = \frac{-1}{2} = -0.5$$
$$\cos 270^\circ = \cos\frac{3\pi}{2} = 0$$
$$\cos 300^\circ = \cos\frac{5\pi}{3} = 0.5$$
$$\cos 330^\circ = \cos\frac{11\pi}{6} = 0.866$$

$$\sin 0^\circ = \sin 360^\circ = \sin 2\pi = 0$$
$$\sin 30^\circ = \sin\frac{\pi}{6} = \frac{1}{2} = 0.5$$
$$\sin 45^\circ = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = 0.707$$
$$\sin 60^\circ = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = 0.866$$
$$\sin 90^\circ = \sin\frac{\pi}{2} = 1$$
$$\sin 120^\circ = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} = 0.866$$
$$\sin 150^\circ = \sin\frac{5\pi}{6} = \frac{1}{2} = 0.5$$
$$\sin 180^\circ = \sin\pi = 0$$
$$\sin 210^\circ = \sin\frac{7\pi}{6} = \frac{-1}{2} = -0.5$$
$$\sin 225^\circ = \sin\frac{5\pi}{4} = \frac{-\sqrt{2}}{2} = -0.707$$
$$\sin 240^\circ = \sin\frac{4\pi}{3} = \frac{-\sqrt{3}}{2} = -0.866$$
$$\sin 270^\circ = \sin\frac{3\pi}{2} = -1$$
$$\sin 300^\circ = \sin\frac{5\pi}{3} = \frac{-\sqrt{3}}{2} = -0.866$$
$$\sin 330^\circ = \sin\frac{11\pi}{6} = \frac{-1}{2} = -0.5$$

Trigonometric reduction formulas

$$\cos(-\theta) = \cos\theta$$

$$\cos(90^\circ + \theta) = -\sin\theta$$

$$\cos(180^\circ - \theta) = -\cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\sin(90^\circ + \theta) = \cos\theta$$

$$\sin(180^\circ - \theta) = \sin\theta$$

$$\tan(90^\circ + \theta) = -\cot\theta$$

$$\tan(180^\circ - \theta) = -\tan\theta$$

Angle sum, angle difference relations

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

Example 4.1

If $\theta = 60°$ and $\phi = 45°$, compute:

a. $\cos 105°$ b. $\sin 15°$ c. $\tan 105°$

Solution:

a. From (4.29), (4.30), (4.43), and (4.44), Page 4–7, we obtain $\cos 45° = \sqrt{2}/2$, $\cos 60° = 1/2$, $\sin 45° = (\sqrt{2})/2$, and $\sin 60° = \sqrt{3}/2$. Then, with these relations and (4.63), Page 4–8, we obtain

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(60° + 45°) = \cos 60° \cdot \cos 45° - \sin 60° \cdot \sin 45°$$

$$\cos(105°) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{1}{4} \cdot (\sqrt{2} - \sqrt{6})$$

$$= \frac{1}{4} \cdot (1.4142 - 2.4495) = \frac{1}{4} \cdot (-1.0353) = -0.2588$$

b. From (4.29), (4.30), (4.43), and (4.44), Page 4–7, we obtain $\cos 45° = \sqrt{2}/2$, $\cos 60° = 1/2$, $\sin 45° = (\sqrt{2})/2$, and $\sin 60° = \sqrt{3}/2$. Then, with these relations and (4.66), Page 4–9, we obtain

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\sin(60° - 45°) = \sin 60° \cdot \cos 45° - \cos 60° \cdot \sin 45°$$

$$\sin(15°) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{1}{4} \cdot (\sqrt{6} - \sqrt{2})$$

$$= \frac{1}{4} \cdot (2.4495 - 1.4142) = \frac{1}{4} \cdot (1.0353) = 0.2588$$

c. From (4.23), Page 4–6,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Then,

$$\tan 105° = \frac{\sin 105°}{\cos 105°}$$

To find the value of $\sin 105°$, we use (4.65), Page 4–8. Thus,

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(60°+45°) = \sin 60°\cdot\cos 45° + \cos 60°\cdot\sin 45°$$

$$\sin(105°) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{4}(\sqrt{6}+\sqrt{2})$$

$$= \frac{1}{4}\cdot(2.4495 + 1.4142) = \frac{1}{4}\cdot(3.8637) = 0.9659$$

We already know the value of $\cos 105°$ from part (a). Therefore,

$$\tan 105° = \frac{\sin 105°}{\cos 105°} = \frac{0.9659}{-0.2588} = -3.7322$$

Fundamental trigonometric identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

Function product relations

$$\cos\theta\cos\phi = \frac{1}{2}\cos(\theta+\phi) + \frac{1}{2}\cos(\theta-\phi)$$

$$\cos\theta\sin\phi = \frac{1}{2}\sin(\theta+\phi) - \frac{1}{2}\sin(\theta-\phi)$$

$$\sin\theta\cos\phi = \frac{1}{2}\sin(\theta+\phi) + \frac{1}{2}\sin(\theta-\phi)$$

$$\sin\theta\sin\phi = \frac{1}{2}\cos(\theta-\phi) - \frac{1}{2}\cos(\theta+\phi)$$

Triangle Formulas

Let Figure 4.6 be any triangle.

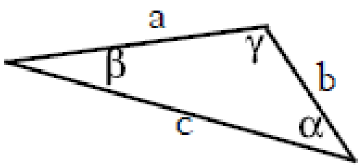


Figure 4.6. Triangle for definition of the laws of sines, cosines, and tangents.

Then,

By the *law of sines:*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

By the *law of cosines:*

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

By the law of tangents:

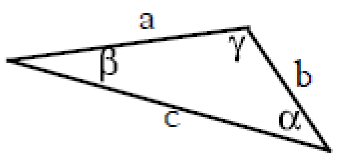
$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)} \qquad \frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)} \qquad \frac{c-a}{c+a} = \frac{\tan\frac{1}{2}(\gamma-\alpha)}{\tan\frac{1}{2}(\gamma+\alpha)}$$

Example 4.2

For the triangle of Figure 4.7below, find the length of side *a* using:

a.the law of sines

b.the law of cosines



$b = 3\ \text{cm} \quad c = 7\text{cm}$

$\alpha = 45° \quad \beta = 25°$

Figure 4.7. Triangle for Example 4.2

Solution:

a. By the law of sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \quad \text{or} \quad \frac{a}{\sin 45°} = \frac{3}{\sin 25°}$$

$\sin 45° = 0.707$ and $\sin 25° = 0.422$

Then $\dfrac{a}{0.707} = \dfrac{3}{0.422}$ and $a = \dfrac{3}{0.422} \times 0.707 = 5.26\ \text{cm}$

b. By the law of cosines,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

or

$$a^2 = 3^2 + 7^2 - 2 \times 3 \times 7\cos 45° = 58 - 42 \times 0.707 = 28.306$$

Therefore,

$$a = 5.32$$

The small difference between the answers in (a) and (b) is due to the rounding of numbers.

Inverse Trigonometric Functions

The notation $\cos^{-1}y$ or arc cos y is used to denote an angle whose cosine is y.

$\therefore$ if $y = \cos x$, then $x = \cos^{-1} y$

Similarly, if $y = \sin x$, then $x = \sin^{-1} y$

If $y = \tan x$, then $x = \tan^{-1} y$

Example 4.3

Find the angle θ if $\cos^{-1} 0.5 = \theta$.

If $\cos^{-1} 0.5 = \theta$, then $0.5 = \cos \theta$

$\cos 60^\circ = 0.5$. Therefore, $\theta = 60^\circ$.

Area of Polygons in Terms of Trigonometric Functions

Area of a triangle

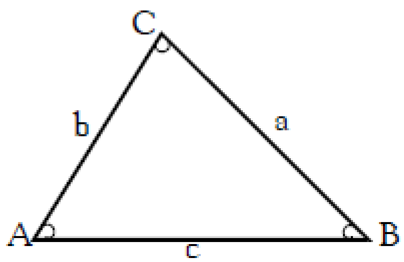


Figure 4.8. General triangle

The area of the general triangle of Figure 4.8 can be found with the formula

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{c^2 \sin A \sin B}{2\sin C}$$

Area of a general quadrilateral

$$\text{Area} = \frac{1}{2}ef\sin\theta = \frac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta$$

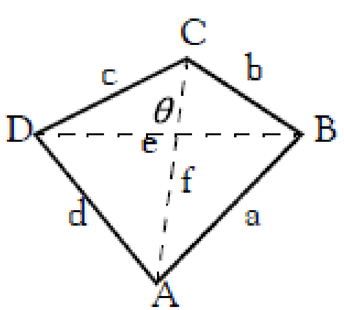


Figure 4.9. General quadrilateral

Area of a parallelogram

$$\text{Area} = ab\sin A = ab\sin B$$

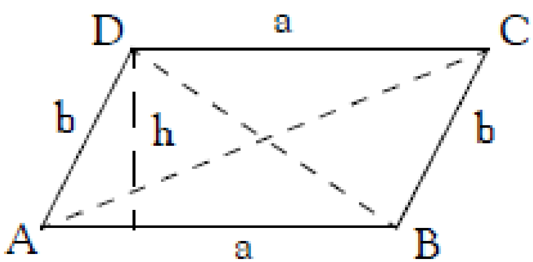


Figure 4.10. Parallelogram

4.10 Exercises

1. If $\theta = 45°$ and $\phi = 30°$, compute:

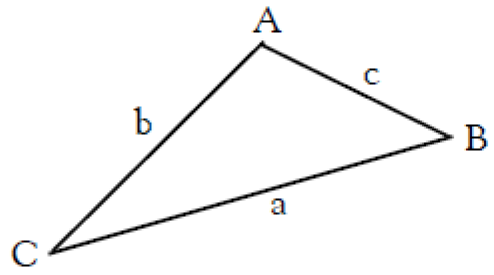
a. $\cos 15°$ b. $\sin 75°$ c. $\tan 75°$

2. Find the angle θ if $\tan^{-1} 1 = \theta$.

3. For the triangle below, find the length of side a using:

a. the law of sines

b. the law of cosines



$b = 4.2$ cm. $c = 2.7$ cm.

$A = 130°$ $C = 20°$